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Moment Methods in Padé Approximation

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The complex sequence $(c_k)_{k=0}^{\infty}$ is assumed to be normal. In the separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$ we solve the "operator moment problem": Find $A \in B(H)$ such that $\langle A^k u_0, u_0 \rangle = c_k$ $(k = 0, 1,...)$, $(u_0 \in H)$. A compact $\Leftrightarrow$ $(c_k)_{k=0}^{\infty}$ has meromorphic generating function $f(z) = \sum_{k=0}^{\infty} c_k z^k$. Padé-type approximants to f, sometimes the ordinary Padé approximants, are obtained as solutions of certain approximate Fredholm equations generated by projections onto finite dimensional subspaces.

INTRODUCTION AND SUMMARY

In 1963 Chisholm [4] showed that Fredholm integral equation techniques can yield convergence results for Padé approximants associated with the Neumann series solution of such an equation. Compare also [5].

Since then, a number of contributions in this direction have appeared. We mention especially Baker's 1975 paper [1], containing many results and applications. In the present paper we exploit the so-called moment method due to Vorobyev [8].

With the Fredholm equation

$$x = zAx + u_0, \tag{1}$$

where A is a compact operator on the separable Hilbert space $(H\langle \cdot, \cdot \rangle)$; x, $u_0 \in H$; $z \in C$, we associate the "approximate equation"

$$x_n = zA_n x_n + u_0,$$

set in the subspace $U_n = \operatorname{span}(u_0, Au_0, A^2u_0,..., A^{n-1}u_0)$ of H. Here A_n: $U_n \to U_n$ is the solution of Vorobyev's moment problem:
"Find an operator A_n: $U_n \to U_n$ satisfying

$$A_n^k u_0 = A^k u_0, \qquad (k = 0, 1,..., n-1),$$
$$A_n^n u_0 = E_n u_n,$$

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where E_n is the orthogonal projection of H onto U_n and $u_n = A^n u_0$." A_n can be extended to all of H by putting $A_n = E_n A E_n$. For (1) we have the solution in the form of a Neumann series,

$$x = u_0 + zAu_0 + z^2 A^2 u_0 + \cdots,$$

valid at least for all $z \in C$ satisfying $|z| < \|A\|^{-1}$.

Putting $\langle A^n u_0, u_0 \rangle = c_n$ $(n = 0, 1,...)$ we have

$$\langle x, u_0 \rangle = \sum_{n=0}^{\infty} c_n z^n = f(z).$$

We show that $\langle x_n, u_0 \rangle$ is the $[n-1/n]$-Padé-type approximant for f.

Vorobyev proves: $x_n \to x$ strongly. This implies a convergence result for these approximants (see Theorem 2.1). If the operator A in (1), apart from being compact, is, moreover, assumed to be simple (see Definition 1.1), then $\langle x_n, u_0 \rangle$ is the $[n-1/n]$-Padé approximant in the ordinary Padé table and we have a convergence result in this case also: Theorem 1.3.

In the case of a non-simple compact operator A, we modify Vorobyev's moment problem in order to obtain results on convergence in the ordinary Padé table. We resort to something akin to oblique projection (see Baker [2]). An important feature of these methods is that we get results on the rate of convergence of $\langle x_n, u_0 \rangle \to \langle x, u_0 \rangle$. (See (1.8).) This brings in the question: Given a power series $\sum_{n=0}^{\infty} c_n z^n$, in advance, can we find a linear operator A (at least bounded) such that $\langle A^n u_0, u_0 \rangle = c_n$ $(n = 0, 1,...)$? This problem, referred to as the "operator moment problem," is treated in Section 4. In the case $\sum_{n=0}^{\infty} c_n z^n$ represents a meromorphic function, which is regular at 0, a compact operator can always be found. Hence the techniques indicated above can sometimes be used as a method in convergence theory of the Padé table. Apart from these applications, the operator moment problem seems to be interesting in its own right.

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Throughout this paper H will denote a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Vorobyev [8] considered the following moment problem:
Given a bounded linear operator A on H and $u_0 \in H$ $(u_0 \neq 0)$, construct a linear operator A_n on the subspace $U_n = \text{span}(u_0, u_1, ..., u_{n-1})$ satisfying

$$\begin{aligned} A_n^k u_0 &= A^k u_0 \qquad (k = 0, 1, ..., n-1), \\ E_n u_n &= E_n A^n u_0 = A_n^n u_0, \end{aligned} \tag{1.1}$$

where E_n is the orthogonal projection onto U_n.

The operator A_n so obtained plays a major role in the convergence theory of the approximations we consider in this paper. We introduce the sequence $(c_n)_{n=0}^{\infty}$ defined by

$$c_n = \langle A^n u_0, u_0 \rangle \qquad (n = 0, 1, \ldots). \tag{1.2}$$

We will assume always that $(c_n)_{n=0}^{\infty}$ is a normal sequence, i.e., all determinants

$$H_n^{(m)} = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-2} \end{vmatrix} \qquad (m, n = 0, 1, \ldots),$$

differ from zero. This has the geometric implication that $\dim U_n = n$ $(n = 0, 1, \ldots)$. This can be proved as follows:

Consider the $(n \times n)$-matrix $(\alpha_{ij})_{i,j=0}^{n-1}$, where

$$\alpha_{ij} = \langle u_i, v_j \rangle \qquad (i, j = 0, 1, \ldots, n-1),$$

and $u_i = A^i u_0$, $v_j = (A^*)^j u_0$ $(i, j = 0, 1, \ldots)$.
Here A^* denotes the adjoint of A. The determinant of this matrix, e.g.,

$$\begin{vmatrix} \langle u_0, v_0 \rangle & \langle u_0, v_1 \rangle & \cdots & \langle u_0, v_{n-1} \rangle \\ \langle u_1, v_0 \rangle & \langle u_1, v_1 \rangle & \cdots & \langle u_1, v_{n-1} \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_{n-1}, v_0 \rangle & \langle u_{n-1}, v_1 \rangle & \cdots & \langle u_{n-1}, v_{n-1} \rangle \end{vmatrix},$$

is

$$H_n^{(0)} = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} \neq 0.$$

If the vectors $u_0, u_1, \ldots, u_{n-1}$ were linearly dependent then the same would hold for the rows of $H_n^{(0)}$. This is impossible.

We put $E_n u_n = -\alpha_0 u_0 - \alpha_1 u_1 - \cdots - \alpha_{n-1} u_{n-1}$, where the α_i are complex numbers. We can write

$$(A_n^n + \alpha_{n-1} A_n^{n-1} + \cdots + \alpha_0 I)\, u_0 = 0.$$

In the following we use the notation $P_n(A_n) = A_n^n + \alpha_{n-1} A_n^{n-1} + \cdots + \alpha_0 I$. In the rest of this section we will consider only those bounded operators $A: H \to H$ that are *simple*.

DEFINITION 1.1. The bounded linear operator A is said to be *simple* if with $u_k = A^k u_0$ $(k = 0, 1,...)$ and $v_j = (A^*)^j u_0$ $(j = 0, 1,...)$

$$\mathrm{span}(u_0, u_1,..., u_{n-1}) = \mathrm{span}(u_0, v_1,..., v_{n-1}), \qquad \forall n \in N,$$

holds.

An important special case of a simple operator is a bounded self-adjoint operator. On the other hand, there exist simple non-self-adjoint operators.

For the coefficients α_i of the polynomial function P_n considered above, a system of equations can be found by observing that $u_n - E_n u_n$ is orthogonal to the space $U_n = \mathrm{span}(u_0, u_1,..., u_{n-1}) = \mathrm{span}(u_0, v_1,..., v_{n-1})$, whence

$$\langle u_n - E_n u_n, v_k \rangle = 0 \qquad (k = 0, 1,..., n-1).$$

Written in full

$$\begin{array}{ccccccccccc}
\langle u_0, v_0\rangle \alpha_0 & + & \langle u_1, v_0\rangle \alpha_1 & +\cdots+ & \langle u_{n-1}, v_0\rangle \alpha_{n-1} & + & \langle u_n, v_0\rangle & = 0 \\
\langle u_0, v_1\rangle \alpha_0 & + & \langle u_1, v_1\rangle \alpha_1 & +\cdots+ & \langle u_{n-1}, v_1\rangle \alpha_{n-1} & + & \langle u_n, v_1\rangle & = 0, \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\langle u_0, v_{n-1}\rangle \alpha_0 & + & \langle u_1, v_{n-1}\rangle \alpha_1 & +\cdots+ & \langle u_{n-1}, v_{n-1}\rangle \alpha_{n-1} & + & \langle u_n, v_{n-1}\rangle & = 0,
\end{array} \tag{1.3}$$

or equivalently

$$\begin{array}{ccccccccc}
c_0\alpha_0 & + & c_1\alpha_1 & +\cdots+ & c_{n-1}\alpha_{n-1} & + & c_n & = & 0\ , \\
c_1\alpha_0 & + & c_2\alpha_1 & +\cdots+ & c_n\alpha_{n-1} & + & c_{n+1} & = & 0\ , \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
c_{n-1}\alpha_0 & + & c_n\alpha_1 & +\cdots+ & c_{2n-2}\alpha_{n-1} & + & c_{2n-1} & = & 0\ .
\end{array} \tag{1.4}$$

We put

$$\Delta_n(z) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix} \qquad (z \in C).$$

See Baker [1, p. 816].

It is easy to verify that

$$z^n P_n(z^{-1}) = \Delta_n(z)/H_n^{(0)}. \tag{1.5}$$

From (1.5) it follows (Perron [7, p. 243] that $z^n P_n(z^{-1})$ is the denominator of the Padé approximant $[n-1/n]_f$ for $\sum_{n=0}^{\infty} c_n z^n = f(z)$.

In the first part of the proof of the following theorem we use an argument due to Brezinski [3, pp. 132–133].

THEOREM 1.1. *Let x_n be the solution of the equation $x_n = zA_n x_n + u_0$ $(x_n \in U_n)$ and z^{-1} not an eigenvalue of A_n which is the solution of the moment problem* (1.1) *for the simple operator A. Let $c_n = \langle A^n u_0, u_0\rangle$ $(n = 0, 1,...)$ and assume that $(c_n)_{n=0}^{\infty}$ is normal. Then $\langle x_n, u_0\rangle = [n-1/n]_f$ for $f(z) = \sum_{n=0}^{\infty} c_n z^n$.*

Proof. P_n denotes the characteristic polynomial of A_n. As $n+1$ elements of U_n are linearly dependent, numbers $\beta_0, \beta_1,..., \beta_n$ exist (not all equal to zero) with

$$\sum_{i=0}^{n} \beta_i A_n^i u_0 = 0,$$

and so

$$\sum_{i=0}^{n} \beta_i A_n^{k+i} u_0 = 0 \qquad (k = 0, 1,...).$$

Put

$$\langle A_n^i u_0, u_0\rangle = c_i^{(n)} \qquad (i = 0, 1,...),$$

to obtain

$$\sum_{i=0}^{n} \beta_i c_{k+i}^{(n)} = 0 \qquad (k = 0, 1,...). \tag{1.6}$$

For a value of z regular with respect to A_n,

$$\langle x_n, u_0\rangle = \langle (I - zA_n)^{-1} u_0, u_0\rangle = \sum_{i=0}^{\infty} c_i^{(n)} z^i.$$

This last series is a recurrent series in view of (1.6) so it is a rational function $\tilde{Q}_{n-1}(z)/\tilde{P}_n(z)$ (indices representing degrees).

Any zero z of $\tilde{P}_n$ makes $I - zA_n$ singular, hence z^{-1} is an eigenvalue of A_n; in other words, z^{-1} is a zero of P_n. Hence $\tilde{P}_n(z)$ is the denominator of $[n-1/n]_f$ for $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We already know $c_i^{(n)} = c_i$ $(i = 0, 1,..., n-1)$. We only have to show that

$$c_i^{(n)} = c_i \qquad (i = n, n+1,..., 2n-1),$$

also holds. Consider the product $(\sum_{i=0}^{\infty} c_i z^i)(z^n P_n(z^{-1}))$. The coefficient of z^k (where $k \in \{n, n+1,..., 2n-1\}$) is

$$\alpha_0 c_{k-n} + \alpha_1 c_{k-n+1} + \cdots + \alpha_{n-1} c_{k-1} + c_k.$$

Then

$$(\alpha_0 c_{k-n} + \alpha_1 c_{k-n+1} + \cdots + \alpha_{n-1} c_{k-1} + c_k)\, H_n^{(0)}$$

is equal to the determinant obtained from $\Delta_n(z)$ upon replacing the last row, i.e., $z^n, z^{n-1},..., 1$ by the row $c_{k-n}, c_{k-n+1},..., c_{k-1}, c_k$. But the determinant so obtained is equal to zero. ∎

Remark 1.1 The coefficient of z^{2n} in the product above is

$$\alpha_0 c_n + \alpha_1 c_{n+1} + \cdots + \alpha_{n-1} c_{2n-1} + c_{2n} = H_{n+1}^0 / H_n^{(0)} \neq 0.$$

We quote the following result from Vorobyev [8].

THEOREM 1.2. *Let $A: H \to H$ be compact and z a regular value for*

$$x = zAx + u_0 \qquad (u_0 \in H). \tag{1.7}$$

Then for $n \in \mathbf{N}$ sufficiently large the equation

$$x_n = zA_n x_n + u_0,$$

where A_n is the solution of the moment problem (1.1), *has a solution $x_n \in U_n = \operatorname{span}(u_0, Au_0,..., A^{n-1}u_0)$ and the sequence (x_n) converges strongly to the solution of* (1.7) (*i.e.*, $\lim_{n\to\infty} \|x_n - x\| = 0$).

The proof is based on the fact that $\lim_{n\to\infty} \|A - A_n\| = 0$. It is important that $\|A_n\| \leqslant \|A\|$.

We restrict ourselves now to the case of a simple compact operator A and to those regular values of z satisfying $|z| < \|A\|^{-1}$. Then the solution of (1.7) is given by the Neumann series

$$x = u_0 + zAu_0 + z^2A^2u_0 + \cdots.$$

Now

$$\langle x, u_0 \rangle = c_0 + c_1 z + c_2 z^2 + \cdots.$$

In view of Theorem 1.1 we have

THEOREM 1.3. *Let A be simple and compact and let the sequence $(c_n)_{n=0}^{\infty}$ of its moments be normal. Then the sequence of $[n-1/n]_f$ Padé approximants to $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges to f for all z satisfying $|z| < \|A\|^{-1}$.*

Remark 1.2. On the rate of convergence the following result (see Vorobyev [8, p. 36, Theorem VII]) is known:
For any real $q > 0$ there exists a positive real number D, not depending on n, such that

$$\|x - x_n\| \leqslant Dq^n.$$

Hence we have for the rate of convergence in our case

$$|\langle x, u_0\rangle - \langle x_n, u_0\rangle| \leqslant \|x - x_n\| \, \|u_0\| \leqslant D \, \|u_0\| \, q^n. \tag{1.8}$$

2.

Let A be a bounded linear operator with a normal sequence of moments $(c_n)_{n=0}^{\infty}$, where $c_n = \langle A^n u_0, u_0\rangle$ $(n = 0, 1,...)$; $u_0 \in H$. The operator A is no longer assumed to be simple. Let $(A_n)_{n=0}^{\infty}$ be the sequence of linear operators solving the moment problem (1.1).

Solving the equation in $U_n = \operatorname{span}(u_0, Au_0,..., A^{n-1}u_0)$,

$$x_n = zA_n x_n + u_0,$$

for any value of z regular with respect to A_n, we have

$$\langle x_n, u_0\rangle = \langle (I - zA_n)^{-1} u_0, u_0\rangle = \sum_{i=0}^{\infty} c_i^{(n)} z^i. \tag{2.1}$$

We still have the relation (1.5), hence

$$\langle x_n, u_0\rangle = R_{n-1}(z)/S_n(z),$$

where R_{n-1} and S_n have degrees $n-1$ and n, respectively. The Taylor expansion of $R_{n-1}(z)/S_n(z)$ agrees with $\sum_{i=0}^{\infty} c_i z^i$ up through the term $c_{n-1} z^{n-1}$. For these "Padé-type" approximants we have

THEOREM 2.1. *Let $A\colon H \to H$ be compact and the sequence $(c_n)_{n=0}^{\infty}$ of its moments be normal. Then the sequence of Padé-type approximants $(R_{n-1}(z)/S_n(z))$ converges to $\sum_{i=0}^{\infty} c_i z^n$ for all z satisfying $|z| < \|A\|^{-1}$.* ∎

Regarding the rate of convergence the same remark can be made as at the end of Section 1.

Remark 2.1. If z is a regular value for A then

$$x = zAx + w$$

has a (unique) solution for every $w \in H$. If we take $w = u_k$ we find, for the Neumann series solution,

$$x = u_k + zAu_k + z^2A^2u_k + \cdots,$$

hence

$$\langle x, u_0 \rangle = \sum_{n=0}^{\infty} c_{k+n} z^n.$$

Hence the result in Theorem 2.1 holds for all series $\sum_{n=0}^{\infty} c_{k+n} z^n$ $(k = 0, 1,...)$.

3.

Let A be a bounded linear operator in H and let $u_0 \in H$ and assume that the moments $c_n = \langle A^n u_0, u_0 \rangle$ $(n = 0, 1, 2,...)$ form a normal sequence. If A is not simple, orthogonal projections no longer lead to ordinary Padé approximants. In order to obtain ordinary Padé approximants we have to resort to oblique projections.

Let $u_n = A^n u_0$ and $v_n = (A^*)^n u_0$, $(n = 0, 1, 2,...)$ and put $U_n = \text{span}(u_0,..., u_{n-1})$ and $V_n = \text{span}(v_0,..., v_{n-1})$. For each n we define a linear mapping $B_n: U_n \to U_n$ by

$$\begin{aligned} B_n u_k &= u_{k+1} \qquad \text{if} \quad k = 0, 1,..., n-2, \\ \text{and } B_n u_{n-1} &= u'_n \qquad \text{if} \quad k = n-1, \end{aligned} \tag{3.1}$$

where

$$u'_n = u_n - P_n(A)\, u_0. \tag{3.2}$$

It is obvious that then $P_n(B_n)\, u_0 = 0$ $(n = 1, 2,...)$. We see that $z^n P_n(z^{-1})$ is the Padé denominator on the field $(n - 1/n)$ in the ordinary Padé table for $\sum_{n=0}^{\infty} c_n z^n$. An analysis as done in Section 1 shows that Theorem 1.1 is a special case of

THEOREM 3.1. *Let A be a bounded linear operator in H and let $u_0 \in H$. Assume that the moments $c_n = \langle A^n u_0, u_0 \rangle$ form a normal sequence and let $U_n = \text{span}(u_0, Au_0,..., A^{n-1}u_0)$ $(n = 1, 2,...)$. Let the linear mapping $B_n: U_n \to U_n$ be defined by* (3.1) *and* (3.2). *Then, if $x_n \in U_n$ satisfies $x_n = zB_n x_n + u_0$ for any regular value of the complex parameter z, then $\langle x_n, u_0 \rangle$ is the $[n - 1/n]$ Padé approximant of the series $\sum_{k=0}^{\infty} c_k z^k$.* ∎

Since $(P_n(A)\,u_0, \bar{P}_n(A^*)\,u_0)_{n=0}^{\infty}$ is a biorthogonal system in H we have $u_n - u_n' \perp V_n$ $(n = 1, 2,...)$. In fact $u_n' = Q_n u_n$, where $Q_n: H \to H$ is the continuous linear projection with kernel $V_n^{\perp}$ and range U_n. (The existence of these projections follows from the normality of $(c_n)_{n=0}^{\infty}$.)

In the case that A is compact and that $(\|Q_n\|)_{n=1}^{\infty}$ is bounded, a modification of Vorobyev's method, i.e., approximation of A by $Q_n A Q_n$, gives similar convergence results of the Padé approximants as in the case that A is simple, Now we drop the assumption that A is compact and that $(\|Q_n\|)_{n=1}^{\infty}$ is bounded, but instead we suppose that $(u_n)_{n=0}^{\infty}$ is a (Schauder) basis of H and that $(S_n)_{n=0}^{\infty}$ is the corresponding sequence of projections (i.e., $S_n(\sum_{k=0}^{\infty} \xi_k u_k) = \sum_{k=0}^{n-1} \xi_k u_k$). Suppose K is the corresponding basis constant, so $K = \sup_n \|S_n\|$. Then we can extend the operators B_n to all of H by

$$B_n u_k = u_{k+1} \qquad \text{for} \quad k = n,\, n+1,... . \tag{3.3}$$

The fact that $(u_n)_{n=0}^{\infty}$ is a basis guarantees us that B_n is continuous. For each $x = \sum_{k=0}^{\infty} \xi_k u_k$ in H we have

$$B_n x = Ax - \xi_{n-1} P_n(A)\, u_0, \tag{3.4}$$

where

$$|\xi_{n-1}| = \frac{\|S_n x - S_{n-1} x\|}{\|u_{n-1}\|} \leqslant \frac{2K}{\|u_{n-1}\|}\,\|x\|,$$

whence

$$\|B_n x\| \leqslant \left(\|A\| + \frac{2K\,\|P_n(A) u_0\|}{\|u_{n-1}\|}\right)\|x\| \tag{3.5}$$

and

$$\|B_n - A\| \leqslant \frac{2K\,\|P_n(A)\,u_0\|}{\|u_{n-1}\|} \qquad (n = 1, 2,...). \tag{3.6}$$

4.

In this section we consider the "operator moment problem": Given a function f, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $c_0 = 1$, can we find a bounded or even compact linear operator A in the separable Hilbert space H such that $\langle A^n u_0, u_0\rangle = c_n$ $(n = 0, 1, 2,...)$, for some $u_0 \in H$?

It turns out that a bounded operator A can be found if the power series $\sum_{n=0}^{\infty} c_n z^n$ has a positive radius of convergence. The reverse also holds. A compact operator can be constructed if and only if f is meromorphic in the whole complex plane and $f(0)=1$.

In the sequel $H=l_2$ and $(e_n)_{n=0}^{\infty}$ is the unit vector basis of H.

LEMMA 4.1. *If $(\rho_n)_{n=0}^{\infty}$ is a sequence of scalars such that $\limsup_n |\rho_n|^{1/n}<\infty$ and $\sup_{k\geqslant n}|\rho_k|>0$, $\forall n$, then there exists a scalar sequence $(\alpha_n)_{n=1}^{\infty}$ which satisfies*

(i) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\rho_n}{\alpha_n}\right|^2<\infty,$

(ii) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\rho_{n+1}}{\alpha_n}\right|^2<\infty$ and

(iii) $\displaystyle\left(\frac{\alpha_{n+1}}{\alpha_n}\right)_{n=1}^{\infty}$ is bounded.

Moreover, if $\limsup_n |\rho_n|^{1/n}=0$, then (iii) can be replaced by

(iv) $\displaystyle\lim_n\frac{\alpha_{n+1}}{\alpha_n}=0.$

Proof. Let $r_n=\sup(|\rho_k|^{1/k}: k\geqslant n)$ and $r=\lim_n r_n$. If we take $\alpha_n=((n^2+1)\,r_n^{2n}-|\rho_n|^2)^{1/2}$, then

$$n^2r_n^{2n}\leqslant\alpha_n^2\leqslant(n^2+1)\,r_n^{2n}\qquad(n=1,2,\ldots).$$

It is easy to verify (i), (ii) and (iii). If $r=0$ then (iv) obviously holds. ∎

THEOREM 4.1. *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ have radius of convergence R. Let $c_0=1$. Then:*

(a) *There exists a bounded linear operator A in H such that $\langle A^ne_0,e_0\rangle=c_n$ $(n=0,1,2,\ldots)$ if and only if $R>0$.*

(b) *If $R=\infty$ then there is a compact linear operator A in H with $\langle A^ne_0,e_0\rangle=c_n$ $(n=0,1,2,\ldots)$.*

Proof. If A is bounded and $\langle A^ne_0,e_0\rangle=c_n$ $(n=0,1,2,\ldots)$, *then* $|c_n|^{1/n}=|\langle A^ne_0,e_0\rangle|^{1/n}\leqslant\|A\|$, so $R>0$.

Conversely let $R>0$. We may assume that $\sup_{k\geqslant n}|\rho_k|>0$, $\forall n$. By Lemma 4.1 there is a sequence of scalars $(\alpha_n)_{n=1}^{\infty}$ such that

(1) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{c_n}{\alpha_n}\right|^2<\infty,$

(2) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{c_{n+1}}{\alpha_n}\right|^2<\infty$ and

(3) $\displaystyle\left(\frac{\alpha_{n+1}}{\alpha_n}\right)_{n=1}^{\infty}$ is bounded if we have only $R>0$, and

(4) $\displaystyle\lim_n \frac{\alpha_{n+1}}{\alpha_n}=0$ if $R=\infty$.

If $Ae_0=c_1e_0+\alpha_1e_1$ and $Ae_n=((c_0c_{n+1}-c_1c_n)/\alpha_n)\,e_0-(\alpha_1c_n/\alpha_n)\,e_1+(\alpha_{n+1}/\alpha_n)\,e_{n+1}$ for $n=1,2,\ldots$, then $A^ne_0=c_ne_0+\alpha_ne_n$ so that $\langle A^ne_0,e_0\rangle=c_n$ for all n and it follows directly from (1), (2), (3) and (4) that A is bounded if $R>0$ and that A is compact if $R=\infty$, since the matrix of A with respect to $(e_n)_{n=0}^{\infty}$ is given by

$$\begin{array}{c|cccccc}
 & e_0 & e_1 & e_2 & \cdots & e_n & \cdots \\ \hline
e_0 & c_1 & \dfrac{d_1}{\alpha_1} & \dfrac{d_2}{\alpha_2} & \cdots & \dfrac{d_n}{\alpha_n} & \cdots \\
e_1 & \alpha_1 & -\dfrac{\alpha_1c_1}{\alpha_1} & -\dfrac{\alpha_1c_2}{\alpha_2} & \cdots & -\dfrac{\alpha_1c_n}{\alpha_n} & \cdots \\
e_2 & 0 & \dfrac{\alpha_2}{\alpha_1} & 0 & \cdots & 0 & \cdots \\
 & \vdots & 0 & \dfrac{\alpha_3}{\alpha_2} & \ddots & \vdots & \\
\vdots & & \vdots & 0 & \ddots & 0 & \\
 & & & & & & \ddots \\
e_{n+1} & & & \vdots & \ddots & \dfrac{\alpha_{n+1}}{\alpha_n} & \\
\vdots & & & & & 0 & \ddots \\
 & & & & & \vdots & \ddots
\end{array}$$

where $d_n=c_0c_{n+1}-c_1c_n$ $(n=1,2,\ldots)$. ∎

Remark 4.1. Let A be the operator constructed in the second part of the proof of Theorem 4.1. If $u_n=A^ne_0$ $(n=0,1,2,\ldots)$ and $\hat{u}_n=u_n/\|u\|$, then the operator $T\colon H\to H$ defined by $Te_n=\hat{u}_n$ $(n=0,1,2,\ldots)$ is an isomorphism of H. This means that $(\hat{u}_n)_{n=0}^{\infty}$ is a basis of H which is equivalent to $(e_n)_{n=0}^{\infty}$. Hence the sequence $(u_n)_{n=0}^{\infty}$ is also a basis. However, if A is compact, then $(u_n)_{n=0}^{\infty}$ cannot be equivalent to $(e_n)_{n=0}^{\infty}$ since in that case the shift operator $e_n\to e_{n+1}$ in l_2 would be compact.

THEOREM 4.2. *Let $\sum_{n=0}^{\infty} c_n z^n$ have a positive radius of convergence and let $c_0 = 1$. Then the following are equivalent:*

(a) *There exists a compact linear operator A in H such that $\langle A^n e_0, e_0 \rangle = c_n$ $(n = 0, 1, 2,...)$.*

(b) *There is a meromorphic function f on $\mathbf{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in some neighborhood of 0.*

Proof. (a) $\Rightarrow$ (b). Let A be compact and suppose that $\langle A^n e_0, e_0 \rangle = c_n$ $(n = 0, 1, 2,...)$. Then the set of singular values for A is at most countable and the only possible point of accumulation is $z = \infty$. Moreover, the singular values $\neq \infty$ for A are poles of the operator-valued function $z \to (I - zA)^{-1}$ which is holomorphic on the set of regular values (see Dunford and Schwartz [6, VII.3.2 and VII.4.5]), so $z \to (I - zA)^{-1}$ is meromorphic on $\mathbf{C}$ and regular at $z = 0$. Consequently $f(z) = \langle (I - zA)^{-1} e_0, e_0 \rangle$ is a scalar-valued meromorphic function on $\mathbf{C}$ which, in addition, satisfies

$$f(z) = \sum_{k=0}^{\infty} \langle z^k A^k e_0, e_0 \rangle = \sum_{k=0}^{\infty} c_k z^k \qquad \text{for} \quad |z| < \|A\|^{-1}.$$

(b) $\Rightarrow$ (a). Let f be meromorphic on $\mathbf{C}$ and let

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \qquad \text{for} \quad |z| < \delta, (\delta > 0).$$

Then there exist entire functions g and h such that

$$f(z) = \frac{1 + zh(z)}{1 - zg(z)} \qquad \text{for all} \quad z \in \mathbf{C} \backslash \{\text{poles}\}. \tag{4.1}$$

Put $h(z) = \sum_{k=0}^{\infty} \mu_k z^k$ and $g(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ for all $z \in \mathbf{C}$. Then it follows from (4.1) that

$$c_{n+1} = \lambda_n c_0 + \lambda_{n-1} c_1 + \cdots + \lambda_0 c_n + \mu_n \qquad (n = 0, 1, 2,...). \tag{4.2}$$

We assume that f is not a rational function. This is certainly the case if the sequence $(c_n)_{n=0}^{\infty}$ is normal. When g and h both reduce to polynomials, a slight modification of the proof given here yields a finite dimensional operator which generates the moments c_n.

Let $u_0 = e_0$, $u_1 = c_1 e_0 + e_1$ and $\alpha_1 = 1$. If $\rho_n = |\lambda_n + \mu_n| + |\mu_{n-1}| \|u_1\|$ $(n = 1, 2,...)$, then $\lim_n |\rho_n|^{1/n} = 0$ and for all n, $\sup_{k \geqslant n} |\rho_k| > 0$. By Lemma 4.1 there is a sequence of scalars $(\alpha_n)_{n=2}^{\infty}$ satisfying

$$\sum_{n=2}^{\infty} \left| \frac{\rho_n}{\alpha_n} \right|^2 < \infty \qquad \text{and} \qquad \lim_n \frac{\alpha_{n+1}}{\alpha_n} = 0. \tag{4.3}$$

We extend u_0, u_1 inductively to a sequence $(u_n)_{n=0}^{\infty}$ by

$$u_{n+1} = (\mu_n + \lambda_n) u_0 + \lambda_{n-1} u_1 + \cdots + \lambda_0 u_n + \alpha_{n+1} e_{n+1} \qquad (n = 1, 2, \ldots) \quad (4.4)$$

and we notice that this relation is also valid for $n = 0$. As $(u_n)_{n=0}^{\infty}$ is an independent sequence we can define the linear operator A in H by

$$Au_n = u_{n+1} \qquad (n = 0, 1, 2, \ldots). \qquad (4.5)$$

Then it follows from (4.4) and (4.5) that

$$\begin{aligned}(\mu_{n-1} + \lambda_{n-1}) u_1 + \lambda_{n-2} u_2 + \cdots + \lambda_0 u_n + \alpha_n A e_n &= u_{n+1} \\ &= (\mu_n + \lambda_n) u_0 + \lambda_{n-1} u_1 + \cdots + \lambda_0 u_n + \alpha_{n+1} e_{n+1} \quad \text{for} \quad n = 1, 2, \ldots,\end{aligned}$$

and this implies

$$Ae_n = \frac{1}{\alpha_n} [(\lambda_n + \mu_n) u_0 - \mu_{n-1} u_1] + \frac{\alpha_{n+1}}{\alpha_n} e_{n+1} \qquad (n = 1, 2, \ldots). \quad (4.6)$$

Together with $Ae_0 = c_1 e_0 + e_1$ this gives, by (4.3) and the definition of ρ_n, that A is compact. Furthermore we have $\langle e_0, e_0 \rangle = 1$, $\langle Ae_0, e_0 \rangle = c_1$ and, by induction, using (4.4) and (4.2),

$$\begin{aligned}\langle A^n e_0, e_0 \rangle &= \langle u_n, u_0 \rangle \\ &= \langle (\mu_{n-1} + \lambda_{n-1}) u_0 + \lambda_{n-2} u_1 + \cdots + \lambda_0 u_{n-1} + \alpha_n e_n, u_0 \rangle \\ &= (\mu_{n-1} + \lambda_{n-1}) c_0 + \lambda_{n-2} c_1 + \cdots + \lambda_0 c_{n-1} = c_n, \quad (n = 2, 3, \ldots). \quad \blacksquare\end{aligned}$$

Remark 4.2. Theorem 4.2 shows that convergence results based on the compactness of the operator are restricted to meromorphic functions.

Remark 4.3. It can be shown that in the above construction we have $u_0 = e_0$ and $u_n = c_n e_0 + \zeta_{n-1} \alpha_1 e_1 + \zeta_{n-2} \alpha_2 e_2 + \cdots + \zeta_0 \alpha_n e_n$ $(n = 1, 2, \ldots)$, where the ζ_n are given by $1/(1 - zg(z)) = \sum_{n=0}^{\infty} \zeta_n z^n$ in some neighborhood of 0. It is easily seen from these relations that in this construction $(u_n)_{n=0}^{\infty}$ is in general not a basis of H.

Remark 4.4. If the elements of the sequence $(c_n)_{n=0}^{\infty}$ are of the special form $c_n = \sum_{k=0}^{\infty} \alpha_k^n |\xi_k|^2$, where $\sum_{k=0}^{\infty} |\xi_k|^2 < \infty$, then the operator moment problem has a very simple solution.

If we define $Ae_n = \alpha_n e_n$ $(n = 0, 1, 2, \ldots)$ and $u_0 = \sum_{k=0}^{\infty} \xi_k e_k$, then $\langle A^n u_0, u_0 \rangle = c_n$ $(n = 0, 1, 2, \ldots)$. Moreover

$$A \text{ is bounded} \Leftrightarrow (\alpha_n)_{n=0}^{\infty} \text{ is bounded}$$

and

$$A \text{ is compact} \Leftrightarrow \lim_n \alpha_n = 0.$$

Baker [1] remarked that the c_n have this special form if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ belongs to a certain class of meromorphic functions.

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